Infinite Regular Hexagon Sequences on a Triangle

Alvy Ray Smith
3420 Cascadia Av S
Seattle WA 98144
alvyray@home.com
16-Nov-99

ABSTRACT. The well-known dual pair of Napoleon equilateral triangles intrinsic to each triangle is extended to infinite sequences of them, shown to be special cases of infinite regular hexagon sequences on each triangle. A set of hexagon-to-hexagon transformations, the hex operators, is defined for this purpose, a set forming an abelian monoid under function composition. The sequences result from arbitrary strings of hex operators applied to a particular truncation of a given triangle to a hexagon. The deep structure of the sequence constructions reveals surprising infinite sequences of non-concentric, symmetric equilateral triangle pairs parallel to one of the sequences of hexagons and provides the most visually striking contribution. Extensive experimentation with a plane geometry educational program inspired all theorems, proofs of which utilize eigenvector analysis of polygons in the complex plane.

INTRODUCTION. This paper is an exercise in the geometry of the complex plane—utilizing the 'eigenpolygon' decomposition of polygons in the complex plane—that extends the well-known pair of Napoleon equilateral triangles intrinsic to each triangle to infinite sequences of them. These sequences, in turn, are special cases of infinite regular hexagon sequences on each triangle.

Another theme is the benefit of experimental use of computer graphics in plane geometry. Geometric constructions in this study are tedious—often infeasible—for the unaided person, yet the intuitions gained from dynamic interaction with the complicated constructions are powerful. Each theorem is the direct result of conjecture inspired by experimentation with normally unwieldy geometric constructions. The software used is an educational program [Sketchpad].

Napoleon's Theorem describes a transformation mapping an arbitrary triangle to an equilateral triangle [Chang-Sederberg 1997; Coxeter-Greitzer 1967; Wetzel 1992]. It is actually a dual pair of transformations leading to the so-called outward and inward Napoleon triangles, called positive and negative here for consistency. Fukuta generalizes the Napoleon transformation to a 2-step transformation that converts an arbitrary triangle to a regular hexagon [Fukuta 1996a; Garfunkel-Stahl 1965; Lossers 1997] and then to a 3-step transformation yielding a different regular hexagon [Chapman 1997; Fukuta 1996b] *strongly* concentric with the first, meaning they are parallel as well (Figure 1). Each transformation is parameterized by σ. At $\sigma = 0$, the first Fukuta hexagon is the positive Napoleon triangle plus its Star-of-David complementary equilateral. Similarly, all hexagon sequences in the paper can be interpreted as equilateral triangle sequences. At $\sigma = 0$, most include one or both Napoleon triangles.

Iteration of the middle step in the 3-step Fukuta transformation is shown to create an infinite sequence of strongly concentric regular hexagons, each being 2 times (the size of) its predecessor (in edge length). The set of two transformations is enlarged to an infinite set by generalizing to what are called the hexagon construction operators, or *hex operators*, and applying them iteratively or in any order to generate infinite sequences of concentric regular hexagons. One such sequence has each hexagon $\sqrt{3}$ times its predecessor and rotated $\frac{\pi}{6}$ from it (Figure 2). Another has each hexagon 2 or 3 times a preceding one and strongly concentric with it (Figure 3). The structure of the transformation set itself is shown to be an abelian monoid in the case of interest.

Finally, the deep structure of the hexagon sequences reveals surprising infinite sequences of non-concentric, symmetric equilateral triangle pairs parallel to one of the sequences of hexagons. Each emerges from a chaos of irregular and regular hexagons (Figure 4) in the most visually interesting contribution of the paper (Figures 5-6).

1. HEX OPERATORS. An arbitrary hexagon of six points $H_1H_2H_3H_4H_5H_6$ is abbreviated H^*, with H_i an arbitrary vertex. Arithmetic on subscripts *i* is modulo 6. Pairs H_iH_{i+3} are the *main diagonals*. 0^* is the degenerate hexagon at the origin.

A *positive* (*negative*) triangle has vertices in counterclockwise (clockwise) order.

Define *positive* n-*interlaced hex operator* $\mathbf{I}_n$ on hexagon H^*: For $n \ge 0$ and all *i*, erect positive equilateral triangle $R_iH_{i+1}H_{i-n}$, a *generating triangle*. Then $\mathbf{I}_nH^*$ is hexagon R^*. *Negative* n-*interlaced hex operator* $\mathbf{i}_n$ is defined similarly but with negative generating triangles $r_iH_{i+1}H_{i-n}$.

Mnemonic names are assigned for $n \le 2$. $\mathbf{P} \equiv \mathbf{I}_0$ and $\mathbf{p} \equiv \mathbf{i}_0$ are the *progressive* hex operators, since each builds equilaterals on successive pairs of vertices. The *nonprogressive* ones are $\mathbf{I} \equiv \mathbf{I}_1$ and $\mathbf{i} \equiv \mathbf{i}_1$, the *interlaced* hex operators, and $\mathbf{B} \equiv \mathbf{I}_2$ and $\mathbf{b} \equiv \mathbf{i}_2$, which are *bi-interlaced*.

Let **F** be the set of hex operators and $\mathbf{F}^+$ the set of nonempty compositions on **F**. These are written as concatenations—eg, $\mathbf{IP}P^*$ means $\mathbf{I}(\mathbf{P}(P^*))$. For empty string ϕ the identity mapping a hexagon to itself, $\mathbf{F}^* = \mathbf{F}^+ \cup \{\phi\}$ is the set of strings of hex operators. The principal purpose here is to generalize the Fukuta (Napoleon) results to infinite sequences of hexagons (equilaterals) on a triangle by exploring the actions of arbitrary strings in $\mathbf{F}^*$.

Define hex operator *iterate* by example: $\mathbf{P}^n$, $n \ge 0$, is defined by $\mathbf{P}^0 = \phi$, $\mathbf{P}^1 = \mathbf{P}$, $\mathbf{P}^{n+1} = \mathbf{PP}^n$.

Define the *successive centroids operator* **C** on a hexagon H^*: Find the centroid C_i of each successive triplet $H_{i-1}H_iH_{i+1}$ of vertices of H^*. Then $\mathbf{C}H^*$ is hexagon C^*.

2. FUKUTA'S PROBLEMS. In arbitrary triangle *ABC*, let P_1 and P_2, P_3 and P_4, P_5 and P_6 be the points on the sides *BC*, *CA*, and *AB* respectively (Figure 1), such that AP_5P_4, P_6BP_1, and P_3P_2C are congruent with one another and similar to *ABC*—ie, P^* is the hexagon obtained by truncating a copy of AP_5P_4 from each vertex of *ABC*. Hence P^* is called a *truncation* of *ABC*, parameterized by $\sigma = |BP_1|/|BC|$, $0 \le \sigma \le 1$. If $\bar{\sigma} = 1 - \sigma$, then $P_1 = \sigma C + \bar{\sigma} B$, and similarly for all P_i. Let *A'*, *B'*, *C'* be the points of intersection of P_1P_4 and P_2P_5, P_3P_6 and P_1P_4, P_2P_5 and P_3P_6, respectively.

[Insert Figure 1 about here.]

Figure 1 shows the results of applying **P** and **IP** to the truncation P^* of triangle *ABC*: $R^* = \mathbf{P}P^*$ and $\mathbf{IP}P^*$ (dashed) are irregular hexagons. Remarkably, both $G_0^* = \mathbf{CP}P^*$ and $G_1^* = \mathbf{CIP}P^*$ (solid bold) are regular hexagons concentric with *ABC*—ie, centered on its centroid. The two hexagons are strongly concentric, and one of them is 2 times the other, and similarly for $g_0^* = \mathbf{Cp}P^*$ and $g_1^* = \mathbf{Cip}P^*$ (solid light). These are superimposed in Figure 1 to show that the two sets of regular hexagons are different, in general, and not strongly concentric with one another. Let ψ denote the angle between positive and negative cases.

There are many other interesting aspects of Fukuta's problems indicated: The main diagonals of $\mathbf{IP}P^*$ are concurrent, equal in length, equally spaced radially, parallel respectively to the sides of $\mathbf{CIP}P^*$ and 3 times their size. So are the main diagonals of $\mathbf{P}P^*$ (relative $\mathbf{CP}P^*$), which are a subset of the $\mathbf{IP}P^*$ main diagonals. Similar results hold for the $\mathbf{ip}P^*$ and $\mathbf{p}P^*$ main diagonals. Furthermore, they intersect the $\mathbf{IP}P^*$ and $\mathbf{P}P^*$ main diagonals at the points *A*', *B*', and *C*'. Regular hexagons $\mathbf{CP}P^*$ and $\mathbf{Cp}P^*$ are strongly concentric if *ABC* is isosceles, as are $\mathbf{CIP}P^*$ and $\mathbf{Cip}P^*$. In this case, $\psi = 0$; generally ψ varies as *ABC* changes but is independent of σ. $\mathbf{CP}P^*$ and $\mathbf{Cp}P^*$ become the same hexagon in the degenerate case of an isosceles triangle with height 0, its base bisected by the third vertex. These other aspects also generalize but, for brevity, will not be further pursued.

3. TERMINOLOGY. For orientation, only the first and second vertex of a hexagon are labeled, generally by 1 and 2. A single shaded triangle *ABC* is used in all figures (to within scaling) for comparison, so redundant labels *A*, *B*, *C*, and P_i are omitted in figures after Figure 1, as are main diagonals and triangle *A*'*B*'*C*'.

All sequences here begin with a construction, the *initialization*, on truncation P^* of *ABC*, which is normally progressive, but nonprogressive initializations are also treated. A typical procedure is: (1) Truncate a triangle to a hexagon. (2) Apply a hex operator to the result of the preceding step. (3) Repeat step 2 $j \geq 0$ times with various hex operators. (4) Apply **C** to the hexagon from step 3 to yield sequence member H_j^*. Step 2 for $j = 0$ is the initialization.

A hex operator applied to a hexagon yields a *generating hexagon*. Use of successive centroids operator **C** on a generating hexagon is a *reduction* of it. In Figure 1, $\mathbf{P}P^*$ is the irregular generating hexagon on truncation P^*, and $\mathbf{CP}P^*$ is the reduction of it to a regular one. In all figures generating hexagons are dashed and reductions of them solid.

It is useful to embed constructions in the complex plane, with origin at the centroid of *ABC*, so $A + B + C = 0$. Thus $P_i + P_{i+2} + P_{i+4} = 0$ by expanding each in terms of σ. The centroid of an arbitrary triangle *PQR* is $\frac{1}{3}(P + Q + R)$. Define operators $\omega = e^{i\pi/3}$ and $\tau = e^{i\pi/6}$, with conjugates $\bar{\omega}$ and $\bar{\tau}$. A *positive regular* hexagon H^* centered on the origin, vertices increasing counterclockwise, has $H_{i+1} = \omega H_i$, and a *negative regular* one $H_{i+1} = \bar{\omega} H_i$. A positive equilateral triangle—eg, $R_1 P_2 P_1$ in Figure 1—is described by $R_1 = \bar{\omega} P_2 + \omega P_1$; a negative one by $r_1 = \omega P_2 + \bar{\omega} P_1$.

4. SAMENESS. Define equivalence relation $\cong$, *sameness*, such that $G^* \cong H^*$ if hexagon G^* is congruent, without rotation or translation, with hexagon H^*. Thus two concentric hexagons are the *same* if they are identical when vertex order and labels are ignored.

Consider hexagon H^* (from a given class of hexagons) and hex operator strings $\mathbf{F}_1, \mathbf{F}_2 \in \mathbf{F}^*$. If $\mathbf{CF}_1 H^* \cong \mathbf{CF}_2 H^*$, then write $\mathbf{F}_1 \cong \mathbf{F}_2$ (for that class). If $\mathbf{CF}_1 H^* \cong s\mathbf{CF}_2 H^*$, s an arbitrary scalar constant, then write $\mathbf{F}_1 \cong s\mathbf{F}_2$. Similarly, if $\mathbf{CF}_1 H^* \cong e^{i\theta}\mathbf{CF}_2 H^*$, write $\mathbf{F}_1 \cong e^{i\theta}\mathbf{F}_2$. For regular hexagons write $\mathbf{F}_1 \cong e^{i\theta \bmod(\pi/3)}\mathbf{F}_2$—eg, $\theta = \pm\frac{\pi}{2}$ implies $\mathbf{F}_1 \cong \tau\mathbf{F}_2$, or $\theta = \pm\frac{n\pi}{3}$, *n* integer, implies $\mathbf{F}_1 \cong \mathbf{F}_2$.

As a first application of sameness, consider the *n*-interlaced hex operators. It is not difficult to see that the positive (negative) 3- and 4-interlaced hex operators are just **i** and **p** (**I** and **P**), respectively, using $\cong$ equivalence. Thus it suffices to restrict $n \leq 2$. Since **B** and **b** are the same, only **B** is studied. Hence sameness collapses the set of hex operators to **F** = {**P**, **I**, **B**, **p**, **i**}.

5. EIGENPOLYGON ANALYSIS. The *shift (backwards) operator* **S** is defined for hexagon G^* by $H^* = \mathbf{S}G^* : H_i = G_{i+1}$ and its inverse $\bar{\mathbf{S}}$ by $H^* = \bar{\mathbf{S}}G^* : H_i = G_{i-1}$. As shown in [Chang-Sederberg 1997], **S** is a linear operator with eigenvalues ω^i, the sixth roots of unity, and eigenvectors $\mathbf{e}^{(i)} = [1 \quad \omega^i \quad \omega^{2i} \quad \omega^{3i} \quad \omega^{4i} \quad \omega^{5i}]$. Similarly, $\bar{\mathbf{S}}$ is a linear operator with the same eigenvectors but with eigenvalues $\bar{\omega}^i$. Any hexagon can be expressed as a complex linear sum of these eigenvectors—hence 'eigenpolygons' or 'basis polygons' [Glassner 1999]. For a polygon with centroid at the origin, as assumed here, the sixth eigenvector is not used.

All hex operators can be written as expressions of the identity and shift operators: $\mathbf{I_n} = \bar{\omega}\mathbf{S} + \omega\bar{\mathbf{S}}^\mathbf{n}$ and $\mathbf{i_n} = \omega\mathbf{S} + \bar{\omega}\bar{\mathbf{S}}^\mathbf{n}$. So can the successive centroids operator: $\mathbf{C} = \frac{1}{3}(\mathbf{S} + \phi + \bar{\mathbf{S}})$. Thus all operators of interest are linear, with the eigenvectors above and easily computed eigenvalues: $\omega^{i-1} + \bar{\omega}^{ni-1}$ and $\omega^{i+1} + \bar{\omega}^{ni+1}$ for the hex operators, respectively, and $\frac{1}{3}(\omega^i + 1 + \bar{\omega}^i)$ for **C**. Let $\lambda_\mathbf{X}$ be the vector of eigenvalues for operator **X**. Then, with simplification,

$$\lambda_\mathbf{P} = [\omega+1 \;\; 2\omega \;\; \omega(\omega+1) \;\; \omega^2 \;\; 0 \;\; 1], \quad \lambda_\mathbf{p} = [0 \;\; -\omega \;\; -\omega(\omega+1) \;\; -2\omega^2 \;\; -\omega^2(\omega+1) \;\; 1]$$

$$\lambda_\mathbf{I} = [2 \;\; 1 \;\; -1 \;\; -2 \;\; -1 \;\; 1], \quad \lambda_\mathbf{i} = [-1 \;\; -2 \;\; -1 \;\; 1 \;\; 2 \;\; 1]$$

$$\lambda_\mathbf{B} = [-\omega^2(\omega+1) \;\; \omega^2 \;\; \omega(\omega+1) \;\; -\omega \;\; -(\omega+1) \;\; 1], \quad \lambda_\mathbf{C} = [\tfrac{2}{3} \;\; 0 \;\; -\tfrac{1}{3} \;\; 0 \;\; \tfrac{2}{3} \;\; 1].$$

Let $\lambda_\mathbf{XY}$ be the vector obtained from pairwise multiplication of $\lambda_\mathbf{X}$ and $\lambda_\mathbf{Y}$—eg,

$$\lambda_\mathbf{CP} = [\tfrac{2}{3}(\omega+1) \;\; 0 \;\; -\tfrac{1}{3}\omega(\omega+1) \;\; 0 \;\; 0 \;\; 1].$$

The eigenpolygon decomposition of truncation P^* has particular importance here. Let E be the 6×6 matrix where each row i is eigenvector $\mathbf{e}^{(i)}$ and each column i is called $\mathbf{E}^{(i)}$. Then $P^* = \mathbf{a}E$ for complex coefficients $\mathbf{a}$. Inverting E yields $\mathbf{a}$ in terms of given parameters:

$$\mathbf{a} = \tfrac{1}{6}[-\bar{\sigma}\omega^2(\omega+1)V \;\;\; \rho\omega^2 v \;\;\; 0 \;\;\; -\rho\omega V \;\;\; \bar{\sigma}(\omega+1)v \;\;\; 0],$$

where $V = -A + \omega B + \bar{\omega}C$, $v = -A + \bar{\omega}B + \omega C$, and $\rho = 1 - 3\sigma$. Then the effect of operator **X** on truncation P^* is computed from $\mathbf{X}P^* = \sum(\lambda_\mathbf{X})_i \mathbf{a}_i \mathbf{e}^{(i)} = \sum \mathbf{a}_{\mathbf{X}_i}\mathbf{e}^{(i)} = \mathbf{a_X} \cdot \mathbf{E}^{(i)}$, where $\mathbf{a_X}$ is the vector obtained from pairwise multiplication of $\lambda_\mathbf{X}$ and **a**. All elements of $\mathbf{a_{CP}}$ are 0 except the first, so $\mathbf{CP}P^*$ must be a regular hexagon, the first eigenpolygon $\mathbf{e}^{(1)}$, with size and orientation given by multiplier $\frac{\bar{\sigma}}{3}V$—ie, $G_0^* = \frac{\bar{\sigma}}{3}V\mathbf{e}^{(1)}$. This is just the solution to Fukuta's first problem. Similar analysis for $\mathbf{CIP}P^*$ shows the same result but with additional multiplier $(\lambda_\mathbf{I})_1 = 2$—ie, $G_1^* = 2\frac{\bar{\sigma}}{3}V\mathbf{e}^{(1)}$, the solution to Fukuta's second problem. The operators annihilate all but one eigenpolygon, which is a regular hexagon. Similarly for the negative Fukuta cases: $g_0^* = \frac{\bar{\sigma}}{3}v\mathbf{e}^{(5)}$, $g_1^* = 2\frac{\bar{\sigma}}{3}v\mathbf{e}^{(5)}$. At $\sigma = 0$, $G_0^* = \frac{1}{3}V\mathbf{e}^{(1)}$ and $g_0^* = \frac{1}{3}v\mathbf{e}^{(5)}$, the *Napoleon hexagons,* include the Napoleon equilaterals.

6. ALGEBRAIC STRUCTURE. The algebraic structure of **F** is established first for hex operators applied to arbitrary hexagons, then refined for application to hexagons that are regular under reduction by **C**. The following useful result can be established straightforwardly:

Lemma 1 (Duality). *For H^* an arbitrary hexagon and h^* the same hexagon with vertices ordered oppositely,* (i) $\mathbf{P}h^* \cong \mathbf{p}H^*$, (ii) $\mathbf{p}h^* \cong \mathbf{P}H^*$, (iii) $\mathbf{I}h^* \cong \mathbf{i}H^*$, (iv) $\mathbf{i}h^* \cong \mathbf{I}H^*$, *and* (v) $\mathbf{B}h^* \cong \mathbf{B}H^*$.

So **P** and **p** are duals on vertex order, as are **I** and **i**; and **B** is self-dual. Thus it suffices to state and prove a theorem for the positive case only.

It is easily checked that all hex operators are commutative and associative, hence:

Lemma 2 (Semigroup). *The hex operators* **F** *under function composition is an abelian semigroup.*

There is now enough machinery to derive the principal tool for special hexagons:

Lemma 3 (Identity). *If* $\mathbf{C}H^*$ *is positive regular, then* $\mathbf{CP}H^*$, $\mathbf{CI}H^*$, $\mathbf{CB}H^* \cong \mathbf{CP}H^*$, $\mathbf{Cp}H^*$, *and* $\mathbf{Ci}H^*$ *are too, with* $\mathbf{P} \cong \sqrt{3}\tau$ *(and* $\mathbf{P}^2 \cong 3$*),* $\mathbf{I} \cong 2$, $\mathbf{B} \cong \mathbf{P}$, $\mathbf{p} \cong 0$, *and* $\mathbf{i} \cong 1$.

Proof. $\mathbf{C}H^*$ positive regular amounts to the requirement that all eigenpolygons in its decomposition be annihilated but the first one. Then clearly $\mathbf{IC}H^*$ simply multiplies the given hexagon by

2—ie, the only operative eigenvalue in $\lambda_{\mathbf{I}}$ is 2—for $\mathbf{IC}H^* = 2\mathbf{C}H^*$. But **I** and **C** commute, so $\mathbf{IC}H^* = \mathbf{CI}H^*$. The other results follow from operative eigenvalues $\omega + 1 = \sqrt{3}\tau$, $\bar{\omega} + 1 = \sqrt{3}\bar{\tau}$, 0, and –1, respectively. ■

The negative case follows by duality: $\mathbf{P} \cong 0$, $\mathbf{I} \cong 1$, $\mathbf{B} \cong \mathbf{p}$, $\mathbf{p} \cong \sqrt{3}\tau$ (and $\mathbf{p}^2 \cong 3$), and $\mathbf{i} \cong 2$ for $\mathbf{C}H^*$ negative regular.

Immediate from **i** being the unique identity for the Lemma 2 semigroup in a special case:

Lemma 4 (Monoid). *The hex operators* **F**, *restricted to hexagons that reduce under* **C** *to positive regular, under function composition form an abelian monoid.*

So, in the monoid, $(\mathbf{F}_1\mathbf{F}_2)\mathbf{F}_3 \cong \mathbf{F}_1(\mathbf{F}_2\mathbf{F}_3)$, $\mathbf{F}_1\mathbf{F}_2 \cong \mathbf{F}_2\mathbf{F}_1$, and $\mathbf{iF_1} \cong \mathbf{F_1 i} \cong \mathbf{F_1}$, for $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3 \in \mathbf{F}^*$. Because $\mathbf{B} \cong \mathbf{P}$, **B** is redundant in the monoid.

The first Fukuta transformation gives $\mathbf{CP}P^*$ positive regular, so iteration of Lemma 3 yields:

Lemma 5 (Progressive Initialization). *For* $\mathbf{F}_1\mathbf{P}P^*$ *a hexagon construction on truncation* P^* *of ABC,* $\mathbf{F}_1 \in \mathbf{F}^*$, *reduction by* **C** *yields a positive regular hexagon—ie,* $\mathbf{CF}_1\mathbf{P}P^*$ *is positive regular.*

Figure 2 shows the result of applying **P** or **p** iteratively to truncation P^*.

[Insert Figure 2 about here.]

Eigenpolygon analysis shows that irregular hexagons result from initialization by **I**, **B**, or **i** instead of **P** or **p**. The following complements Lemma 5 by stating when a construction on a nonprogressive initialization may too yield a regular hexagon.

Lemma 6 (Nonprogressive Initialization). *For* $\mathbf{F}_1\mathbf{I}P^*$ *(*$\mathbf{F}_1\mathbf{B}P^*$*) on truncation* P^*, $\mathbf{F}_1 \in \mathbf{F}^*$, *reduction by* **CP** *(instead of* **C***) yields a positive regular hexagon—ie,* $\mathbf{CPF}_1\mathbf{I}P^*$ *(*$\mathbf{CPF}_1\mathbf{B}P^*$*) is positive regular.*

Proof. By commutativity, $\mathbf{CPF}_1\mathbf{I}P^* = \mathbf{CF}_1\mathbf{IP}P^*$, positive regular by Lemma 5. Similarly for **B**. ■

Hexagon sequences generated by strings $\mathbf{F}_1$ containing no operators **P** or **p** can be shown irregular by eigenpolygon analysis, so Lemma 6 states that no new regular hexagons result from nonprogressive initializations. It also implies that each regular hexagon sequence can be generated in a different way—eg, in place of construction $\mathbf{I}^a\mathbf{P}^b\mathbf{P}P^*$ reduced by **C**, use construction $\mathbf{I}^a\mathbf{B}^bP^*$ reduced by **CP** to the same regular hexagon. $\mathbf{B} \cong \mathbf{P}$ in this case only after an application of at least one **P**, hence **B** is not redundant in **F**, in general, although it is in the monoid.

7. INFINITE REGULAR HEXAGON SEQUENCES. The most general regular hexagon sequences due to the hex operators are described next, where H^* a *2-3 multiple* of G^* means $H^* \cong 2^m 3^n G^*$, $m \geq 0$, $n \geq 0$, and where *disjoint* means disjoint as sets not as geometry.

Theorem 1. *For arbitrary triangle ABC with truncation* P^*, *there exist these infinite, concentric but disjoint, positive regular hexagon sequences centered on ABC, generated by strings in* $\mathbf{F}^*$ *for* $j \geq 0$:

(i) $\boldsymbol{S} = \{S_j^* : S_0^* = \mathbf{CP}P^*,\ S_j^*$ *a 2-3 multiple of* $S_0^* = G_0^*\}$,

(ii) $\boldsymbol{S}_\tau = \{S_{\tau_j}^* : S_{\tau_0}^* = \mathbf{CP}^2P^*,\ S_{\tau_j}^*$ *a 2-3 multiple of* $S_{\tau_0}^* = H_1^*\}$.

Proof. By Lemma 3, iterates of **I** and $\mathbf{P}^2$ multiply a positive regular hexagon by powers of 2 or 3, respectively, into strongly concentric positive regular hexagons of larger size. 2-3 multiples can be ordered uniquely by numeric size. Suppose $2^m 3^n$ is the next factor in succession. Then $S_j^* = \mathbf{CI}^m(\mathbf{P}^2)^n\mathbf{P}P^*$ generates the corresponding element of $\boldsymbol{S}$ and is positive regular. $\boldsymbol{S}_\tau$ is generated the same way as $\boldsymbol{S}$ but with one extra application of **P** which rotates the sequence $\frac{\pi}{6}$ from $\boldsymbol{S}$. That is, $S_{\tau_j}^* = \mathbf{CI}^m(\mathbf{P}^2)^n\mathbf{P}^2P^* \cong \sqrt{\mathbf{3}}\tau S_j^*$. Each of $\boldsymbol{S}$ and $\boldsymbol{S}_\tau$ is strongly concentric. ■

Figure 3 shows the first seven elements of $\boldsymbol{S}$ and five of $\boldsymbol{S}_\tau$. The $\boldsymbol{S}$ elements are obtained by applying $\mathbf{I}^0$, $\mathbf{I}^1$, $\mathbf{P}^2$, $\mathbf{I}^2$, $\mathbf{I}^1\mathbf{P}^2$ (or its commuted equivalents), $\mathbf{I}^3$, and $\mathbf{P}^4$, respectively, to $\mathbf{P}P^*$. Let $\boldsymbol{s}$ and $\boldsymbol{s}_\tau$ be the duals to $\boldsymbol{S}$ and $\boldsymbol{S}_\tau$. $\boldsymbol{S}_\tau$ and $\boldsymbol{s}_\tau$ do not contain Napoleon hexagons at $\sigma = 0$.

[Insert Figure 3 about here.]

Theorem 2. *Concentric sequence* $\boldsymbol{A} = \{\boldsymbol{S}\} \cup \{\boldsymbol{s}\} \cup \{\boldsymbol{S}_\tau\} \cup \{\boldsymbol{s}_\tau\} \cup \{0^*\}$ *is a disjoint union, in general, containing every unique regular hexagon (by* $\cong$ *equivalence) generable by strings in* $\mathbf{F}^*$.

Proof. Let $\boldsymbol{A}$ be the ordered union of the two sequences of Theorem 1, the two from its dual, and the one element $A_0^* = 0^*$ absent from all of them. These are disjoint sets, except in the degenerate case of *ABC* an isosceles triangle of height 0, mentioned earlier, when $\boldsymbol{S}$ and $\boldsymbol{s}$ ($\boldsymbol{S}_\tau$ and $\boldsymbol{s}_\tau$) are the same; but $\boldsymbol{A}$ is never strongly concentric. Because of commutativity and Lemma 6 (and its dual), it suffices to consider only constructions of the form $\mathbf{C}\mathbf{I}^a\mathbf{P}^b\mathbf{i}^c\mathbf{p}^d H^*$ on the initializations $H^* \in \{\mathbf{P}P^*,\ \mathbf{p}P^*\}$, for non-negative integers *a*, *b*, *c*, and *d*. $\boldsymbol{S}$ and $\boldsymbol{S}_\tau$ exhaust all cases for which $c = d = 0$ on $H^* = \mathbf{P}P^*$. For this initialization, $\mathbf{i}^c$, $c > 0$, is an identity creating no new hexagons, and $\mathbf{p}^d$, $d > 0$, always zeroes to 0^*. The negative case follows by duality, so $\boldsymbol{A}$ exhausts $\mathbf{F}^*$. ■

Simply iterating the elements of $\mathbf{F}$ generates useful infinite regular subsequences of $\boldsymbol{A}$. The following is easily established by Lemma 3 and simple induction.

Theorem 3. *For arbitrary triangle ABC with truncation* P^**, there exist these infinite concentric positive regular hexagon sequences centered on ABC, generated respectively by iterates of* $\mathbf{I}$ *and* $\mathbf{P}$ for $j \geq 0$:

(i) $\boldsymbol{G} = \{G_j^*:\ G_0^* = \mathbf{C}\mathbf{P}P^*,\ G_{j+1}^*$ *2 times* $G_j^*\}$,

(ii) $\boldsymbol{H} = \{H_j^*:\ H_0^* = \mathbf{C}\mathbf{P}P^*,\ H_{j+1}^*\ \sqrt{3}$ *times* H_j^* *and rotated* $\frac{\pi}{6}\}$.

Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be the dual sequences generated by iterates of $\mathbf{i}$ and $\mathbf{p}$. The first two elements of $\boldsymbol{G}$ and $\boldsymbol{g}$ ($\boldsymbol{H}$ and $\boldsymbol{h}$) are shown in Figure 1 (Figure 2). Fukuta's two positive-case (negative-case) hexagons are the first two in sequence $\boldsymbol{G}$ ($\boldsymbol{g}$), hence $\boldsymbol{S}$ ($\boldsymbol{s}$). $\boldsymbol{G}$ is strongly concentric and a subset of $\boldsymbol{S}$. The even elements of $\boldsymbol{H}$ are also a subset of $\boldsymbol{S}$, but the odd ones belong to $\boldsymbol{S}_\tau$.

8. DEEP STRUCTURE. Although identities $\mathbf{i}^m$ contribute nothing to hex operator constructions reduced by $\mathbf{C}$, they do induce an interesting regular structure in general. Let $U_m^* = \mathbf{i}^m\mathbf{P}P^*$ be the generating hexagon for $\mathbf{i}^m$. Figure 4 shows that the *identity cluster* $\{U_m^*,\ m \geq 0\}$ of distinct generating hexagons all map to one regular hexagon $\mathbf{C}\mathbf{P}P^*$. Nevertheless, personal computer experiments [Sketchpad] strongly suggest the sceptre structure defined in the lemma below, where a *sceptre* (from the acronym of "symmetric, congruent, equilateral, parallel triangles") is always formed by the intersections of two triples of concurrent equiangular lines, with the lines of one triple pairwise parallel those of the other (Figure 4).

[Insert Figure 4 about here.]

Lemma 7 (Identity Cluster). *The identity cluster on truncation* P^* *of ABC has these properties*:

(i) *For each i, vertices labeled i, m even, and i+3, m odd, form line* L_i *parallel a side of hexagons* $\boldsymbol{g}$. *Call* L_i *a vertex locus.*

(ii) *At* $\sigma = \frac{1}{3}$*, every hexagon* U_m^* *is the same as the Napoleon hexagon, strongly concentric with its reduction by* $\mathbf{C}$*, and* $\frac{3}{2}$ *its size—ie,* $U_m^* \cong \frac{3}{2}\mathbf{C}U_m^*$.

(iii) L_i, L_{i+2}, L_{i+4} *are concurrent. Let* $\mathbf{L_1}$ *be the triple for* $i = 1$, $\mathbf{L_2}$ *for* $i = 2$*, and* $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *be the corresponding points of concurrency.*

(iv) *The elements of* $\mathbf{L_2}$ *intersect C, A, and B, respectively*.

(v) *The intersection points of* $\mathbf{L_1}$ *and* $\mathbf{L_2}$ *form a sceptre—ie, a pair of congruent parallel equilateral triangles* $\Delta_1 = \mathbf{C}_1T_3T_4$, $\Delta_2 = \mathbf{C}_2T_6T_1$, *with* $T_i = L_i \cap L_{i+1}$*—parallel hexagons* ***g***.

(vi) $\mathbf{C}_1\mathbf{C}_2$ *is collinear the centroid of ABC;* $|\mathbf{C_1C_2}|$ *is the length of the identity cluster main diagonals.*

(vii) *At* $\sigma = \frac{1}{3}$, $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *are symmetric about the origin. At* $\sigma = \frac{2}{3}$, $\mathbf{C_1}$ *or* $\mathbf{C_2}$ *is coincident with the origin. At* $\sigma = 1$, $\mathbf{C_1}$ *and* $\mathbf{C_2}$ *are coincident.*

(viii) $T_6T_1T_3T_4$ *is a parallelogram with angle* ψ, $T_6T_1 \in \Delta_1$, $T_3T_4 \in \Delta_2$, $|T_1T_3| = |\mathbf{C_1C_2}|$, *and* $T_1T_3T_5$, $T_2T_4T_6$ *are a pair of congruent equilaterals (a sceptre) parallel hexagons* ***G***, *with sides* 3 *times a side of* G_0^*.

Proof. (i) $U_m^* = (-1)^m sV\mathbf{e}^{(1)} + (-2)^{m+1}rv\mathbf{e}^{(2)} + rV\mathbf{e}^{(4)}$, with $r = \frac{\rho}{6}$, $s = \frac{\bar{\sigma}}{2}$. Experiment suggests $U_m^* - \mathbf{S}^3U_{m-1}^*$ are vertex loci. Indeed, $U_m^* - \mathbf{S}^3U_{m-1}^* = 2(-2)^{m-1}\rho v\mathbf{e}^{(2)}$, $m \ge 1$, and induction on m proves that all these lines must pass through U_0^*. So L_i has direction $v\mathbf{e}_i^{(2)}$, parallel sides of hexagons ***g***.

(ii) $\sigma = \frac{1}{3}$ gives $r = 0$, $s = \frac{1}{3}$, and $U_m^* = \frac{1}{3}(-1)^m V\mathbf{e}^{(1)}$, a regular hexagon the same as the Napoleon hexagon, which reduces to $\mathbf{C}U_m^* = \frac{2}{9}(-1)^m V\mathbf{e}^{(1)} \cong G_0^*$.

(iii) Without loss of generality, the equations for L_i can be computed from U_2^* and U_0^* as equations $(\bar{U}_2^* - \bar{U}_0^*)z - (U_2^* - U_0^*)\bar{z} + U_2^*\bar{U}_0^* - \bar{U}_2^*U_0^* = 0$. From (i), $U_2^* - U_0^* = -\rho v\mathbf{e}^{(2)}$, so they are $-\bar{v}\mathbf{e}^{(4)}z + v\mathbf{e}^{(2)}\bar{z} + \bar{v}V(s\mathbf{e}^{(5)} + r\mathbf{e}^{(2)}) - v\bar{V}(s\mathbf{e}^{(1)} + r\mathbf{e}^{(4)}) = 0$. The systems of equations for $\mathbf{L_1}$, $\mathbf{L_2}$ are

$$\begin{bmatrix} \bar{v} & -v & a(-\bar{v}V + v\bar{V}) \\ \bar{v}\omega & v & a(\bar{v}V + v\bar{V}\omega) \\ \bar{v} & v\omega & a(\bar{v}V\omega + v\bar{V}) \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \\ 1 \end{bmatrix} = 0, \quad \begin{bmatrix} \bar{v} & v\omega & b(-\bar{v}V\omega - v\bar{V}) \\ \bar{v} & -v & b(\bar{v}V - v\bar{V}) \\ \bar{v}\omega & v & b(-\bar{v}V - v\bar{V}\omega) \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \\ 1 \end{bmatrix} = 0,$$

$a = s + r$, $b = s - r$. Both determinants vanish, so $\mathbf{L_1}$, $\mathbf{L_2}$ are concurrent, respectively. Let $\mathbf{u}_v$ be the unit vector for v. Solving gives $\mathbf{C}_1 = -a\mathbf{u}_v^2\bar{V}$, $\mathbf{C}_2 = b\mathbf{u}_v^2\bar{V}$. So $\mathbf{C}_1\mathbf{C}_2 = \mathbf{C}_2 - \mathbf{C}_1 = \bar{\sigma}\mathbf{u}_\mathbf{v}^2\bar{V}$.

(iv) $U_0^* = sV\mathbf{e}^{(1)} - 2rv\mathbf{e}^{(2)} + rV\mathbf{e}^{(4)}$. $V + v = -3A$ and $s - r = \frac{1}{3}$, so $U_{0_4} - A = \sigma v$ is parallel L_4 through point U_{0_4}, and A must lie on L_4. Similarly for $U_{0_6} - B = -\omega\sigma v$, $U_{0_2} - C = \omega^2\sigma v$.

(v) A main diagonal of each generating hexagon connects a point on a vertex locus in $\mathbf{L_1}$ to another in $\mathbf{L_2}$. Thus $\mathbf{C_1}$ and $\mathbf{C_2}$ are not the same in general. Since the L_i parallel the sides of hexagons ***g***, so must the equilateral triangles of the sceptre.

(vi) The equations in (iii) for $\mathbf{C_1}$, $\mathbf{C_2}$ show them collinear the origin, and $|\mathbf{C}_1\mathbf{C}_2| = \bar{\sigma}|V|$, the length of the main diagonals of the identity cluster.

(vii) These special cases are readily derived from the formulas in (iii) for $\mathbf{C_1}$, $\mathbf{C_2}$.

(viii) $|\mathbf{C}_1\mathbf{C}_2|$ is 3 times the side of G_0^* since it has the same length as a main diagonal of the identity cluster. Compute T_1, T_3 using equations in (iii) and the difference $T_1 - T_3 = \bar{\sigma}V$. ■

A sceptre is rotationally symmetric and has six equal sides and angles, so is, in a sense, a regular hexagon—albeit a disconnected one. The next theorem establishes infinite sequences of them too. Let $T^\Delta = \Delta_1\Delta_2 = \mathbf{C}_1T_3T_4\mathbf{C}_2T_6T_1$ denote a sceptre, where the labeling of Lemma 7 is used for right-handed sceptres in general (see left-handed sceptre in Figure 6). Call line segment $\mathbf{C_1C_2}$ the *main diagonal* of a sceptre. In general, a sceptre T^Δ is not centered at the origin. Let oT^Δ be its offset, and Δ_kT^Δ be its equilateral triangle Δ_k, $k = 1$ or 2.

Let *sceptre constructor* $\mathbf{\Psi}(H^*)$ be defined on hexagons H^* that are hex operator constructions on $\mathbf{P}P^*$. $\boldsymbol{\psi}(H^*)$ is the corresponding dual sceptre constructor on $\mathbf{p}P^*$. $\mathbf{\Psi}(H^*)$ is constructed, without loss of generality, as follows: (1) Form directed line segments L_i from, say, $(\mathbf{i} - \mathbf{S}^3)H^*$ or $(\mathbf{i}^2 - \phi)H^*$. (2) If the L_i meet the conditions defining a sceptre, return the sceptre, else $\mathbf{\Psi}(H^*)$ is undefined. The next lemma establishes that $\mathbf{\Psi}(H^*)$ is defined for all hexagons in its domain. Let E_0^Δ be the Lemma 7 sceptre, but centered at the origin, and let $\mathbf{r} = r\mathbf{u}_\mathbf{v}^2\bar{V}$, for which $|\mathbf{r}| = 0$ at $\sigma = \frac{1}{3}$. Then $T_0^\Delta = \mathbf{\Psi}(\mathbf{P}P^*) = E_0^\Delta - \mathbf{r}$, so $T_0^\Delta - oT_0^\Delta = E_0^\Delta$ and $oT_0^\Delta = -\mathbf{r}$. Let 0^Δ be the sceptre of size 0 at the origin. Extend sameness and 2-3 multiples to sceptres, and arbitrary hex operator $\mathbf{X}$ to sceptres by $\mathbf{X}[\mathbf{\Psi}(H^*)] = \mathbf{\Psi}(\mathbf{X}H^*)$.

Lemma 8 (Sceptre Operators). *For sceptre T^Δ from hex operators on $\mathbf{P}P^*$ of triangle ABC,*

(i) $\mathbf{I}[T^\Delta] \cong -2T^\Delta$, $\mathbf{B}[T^\Delta] \cong \mathbf{P}[T^\Delta]$, $\mathbf{p}[T^\Delta] = 0^\Delta - 2oT^\Delta$, $\mathbf{i}[T^\Delta] \cong T^\Delta$, *and* $\mathbf{P}^2[T^\Delta] = -3(T^\Delta - oT^\Delta) + oT^\Delta$ *are sceptres;*

(ii) $\mathbf{P}[T^\Delta]$ *is a sceptre with main diagonal* $\sqrt{3}$ *that of* T^Δ *and orthogonal to it,* $\Delta_k\mathbf{P}[T^\Delta]$ $\sqrt{3}$ *times* $\Delta_k T^\Delta$, $k = 1$ *or 2, and* $o\mathbf{P}[T^\Delta] = oT^\Delta$*;*

(iii) $\Delta_k T^\Delta$ *reflected about one of its sides is coincident with* $\Delta_k\mathbf{P}[T^\Delta]$, $k = 1$ *or 2—ie, two vertices of* $\Delta_k T^\Delta$ *are collinear two sides of* $\Delta_k\mathbf{P}[T^\Delta]$.

Proof. The general case identity cluster is, without loss of generality, $W_m^* = \mathbf{i}^m\mathbf{I}^a\mathbf{P}^b\mathbf{i}^c\mathbf{p}^d\mathbf{B}^e\mathbf{P}P^*$. $W_2^* - W_0^* = (-1)^{c+d+1}2^{b+c}\omega^{b+d+2e}\rho v\mathbf{e}^{(2)}$, so vertex loci L_i exist with the same orientations as in Lemma 7. The systems of equations for $\mathbf{L_1}$, $\mathbf{L_2}$ are derived and solved as there. The solution is $\mathbf{C}_1\mathbf{C}_2 = (-1)^{c+e}2^a 0^d\omega^{b+e}(\omega+1)^{b+e}\bar{\sigma}\mathbf{u}_\mathbf{v}^2\bar{V}$ with midpoint at $(-1)^{a+d+1}2^{a+d}\mathbf{r}$. This is sufficient to establish (i) and (ii). (iii) is proved by showing that loci L_i and L_{i+1} of $\Delta_1 T^\Delta$ are concurrent with locus L_i of $\Delta_1\mathbf{P}[T^\Delta]$, $i = 3$ and 4, calculations as in the concurrency proofs above. ■

So $\mathbf{B}$ is redundant here and even one application of $\mathbf{p}$ annihilates a sceptre. Note that $\mathbf{P}$ swaps the handedness of a sceptre (Figure 6). Infinite sceptre sequences follow immediately from the lemma for any regular hexagon sequence. They can be thought of as the regular structures lying "between" the hexagons—constructionally, not spatially—created by one or more identity operators $\mathbf{i}$ applied there.

Theorem 5. *Each infinite regular hexagon sequence* $\boldsymbol{Q}$ *on truncation* P^* *of triangle ABC has a corresponding infinite sceptre sequence* $\mathbf{\Psi}(\boldsymbol{Q}) = \{\mathbf{\Psi}(Q_j) : Q_j \in \boldsymbol{Q}\}$. *In particular, for* $j > 0$,

(i) $\mathbf{\Psi}(\boldsymbol{G})$ *has sceptres parallel* $\boldsymbol{g}$ *and one another, strongly concentric at* $\sigma = \frac{1}{3}$, *diagonals collinear one another and the centroid of ABC, and* $\mathbf{\Psi}(G_j)$ *2 times* $\mathbf{\Psi}(G_{j-1})$;

(ii) $\mathbf{\Psi}(\boldsymbol{H})$ *has sceptres parallel* $\boldsymbol{g}$*, diagonal of* $\mathbf{\Psi}(H_j)$ *perpendicular that of* $\mathbf{\Psi}(H_{j-1})$ *and concentric with it, and* $\Delta_k\mathbf{\Psi}(H_j)$ $\sqrt{3}$ *times* $\Delta_k\mathbf{\Psi}(H_{j-1})$;

(iii) $\mathbf{\Psi}(\boldsymbol{S})$ *has sceptres parallel* $\boldsymbol{g}$ *and one another, strongly concentric at* $\sigma = \frac{1}{3}$, *diagonals collinear one another and the centroid of ABC, and* $\mathbf{\Psi}(S_j)$ *a 2-3 multiple of* $\mathbf{\Psi}(G_0)$, *to within translation;*

(iv) $\mathbf{\Psi}(\boldsymbol{S}_\tau)$ *has sceptres parallel* $\boldsymbol{g}$ *and one another, strongly concentric at* $\sigma = \frac{1}{3}$, *diagonals collinear one another and the centroid of ABC, and* $\mathbf{\Psi}(S_{\tau_j})$ *a 2-3 multiple of* $\mathbf{\Psi}(H_1)$, *to within translation;*

(v) $\mathbf{\Psi}(\boldsymbol{A}) = \mathbf{\Psi}(\boldsymbol{S}) \cup \boldsymbol{\psi}(\boldsymbol{s}) \cup \mathbf{\Psi}(\boldsymbol{S}_\tau) \cup \boldsymbol{\psi}(\boldsymbol{s}_\tau) \cup \{0^\Delta\}$ *is the disjoint set of all possible sceptres from hex operators on* P^* *of ABC, to within translation.*

Figures 5-6 show the first two elements of $\mathbf{\Psi}(\boldsymbol{G})$ (hence $\mathbf{\Psi}(\boldsymbol{S})$) and $\mathbf{\Psi}(\boldsymbol{H})$, two of these remarkable infinite structures, and the corresponding hexagon sequences. The computer is invaluable for graphic study of these complex structures and how they change dynamically with σ. Small changes in σ away from the value shown lead to complicated diagrams. The structures in the theorem hold, of course, but the generating hexagons become nonconvex and self-intersecting in irregular ways. Interaction with a diagram that varies with σ allows one to experience the sudden appearance, near $\sigma = \frac{1}{3}$, of the Figures 4-6 configurations from a chaos of lines and then their abrupt disappearance back to an unreadable complexity of lines.

[Insert Figures 5-6 about here.]

REFERENCES

[Chang-Sederberg 1997] G. Chang and T. W. Sederberg, *Over and Over Again*, Mathematical Association of America, Washington, DC, 1997, 57-61, 90-98.

[Chapman 1997] R. Chapman, A regular hexagon emerging from a triangle (Solution to problem proposal 10514), *The American Mathematical Monthly* **104**(1997), 75.

[Coxeter-Greitzer 1967] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, DC, 1967, 60-65.

[Fukuta 1996a] J. Fukuta, Problem proposal 1493, *Mathematics Magazine* **69**(1996), 67.

[Fukuta 1996b] J. Fukuta, Problem proposal 10514, *The American Mathematical Monthly* **103**(1996), 267-268.

[Garfunkel-Stahl 1965] J. Garfunkel and S. Stahl, The triangle reinvestigated, *The American Mathematical Monthly* **72**(1965), 12-20.

[Sketchpad] *The Geometer's Sketchpad 3*, Key Curriculum Press, P O Box 2304, Berkeley, CA 94702-0304, 1-800-995-MATH, http://www.keypress.com.

[Glassner 1999] A. Glassner, Fourier polygons, *IEEE Computer Graphics and Appications* **19**(1999), 84-91.

[Lossers 1997] O. P. Lossers, A generalization of Napoleon's theorem (Solution to problem proposal 1493), *Mathematics Magazine* **70**(1997), 70-73.

[Wetzel 1992] J. E. Wetzel, Converses of Napoleon's theorem, *The American Mathematical Monthly* **99**(1992), 339-351.

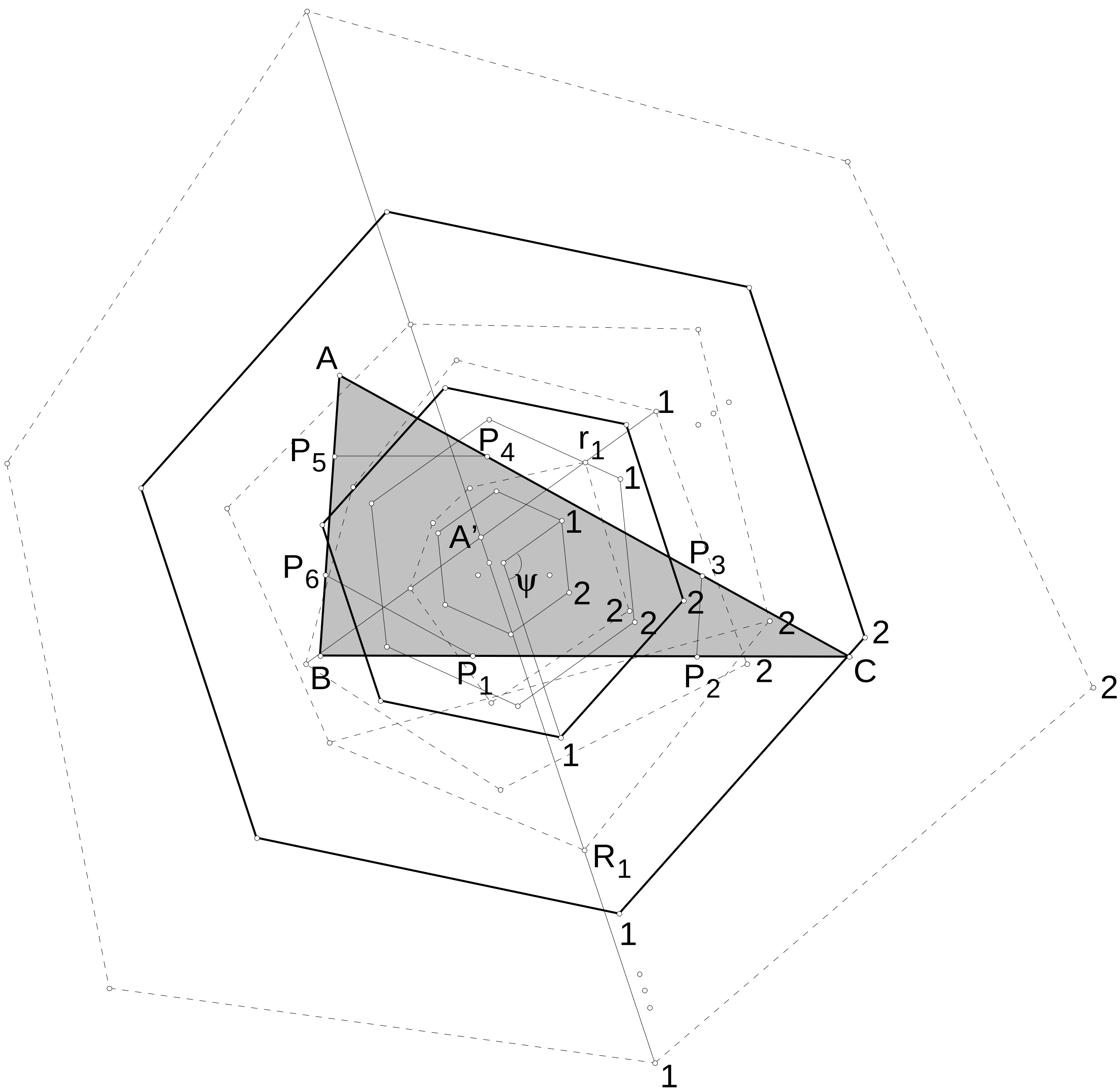

FIGURE 1

First two elements of sequences ***G*** (bold) and ***g*** (light), hence ***S*** and ***s***.

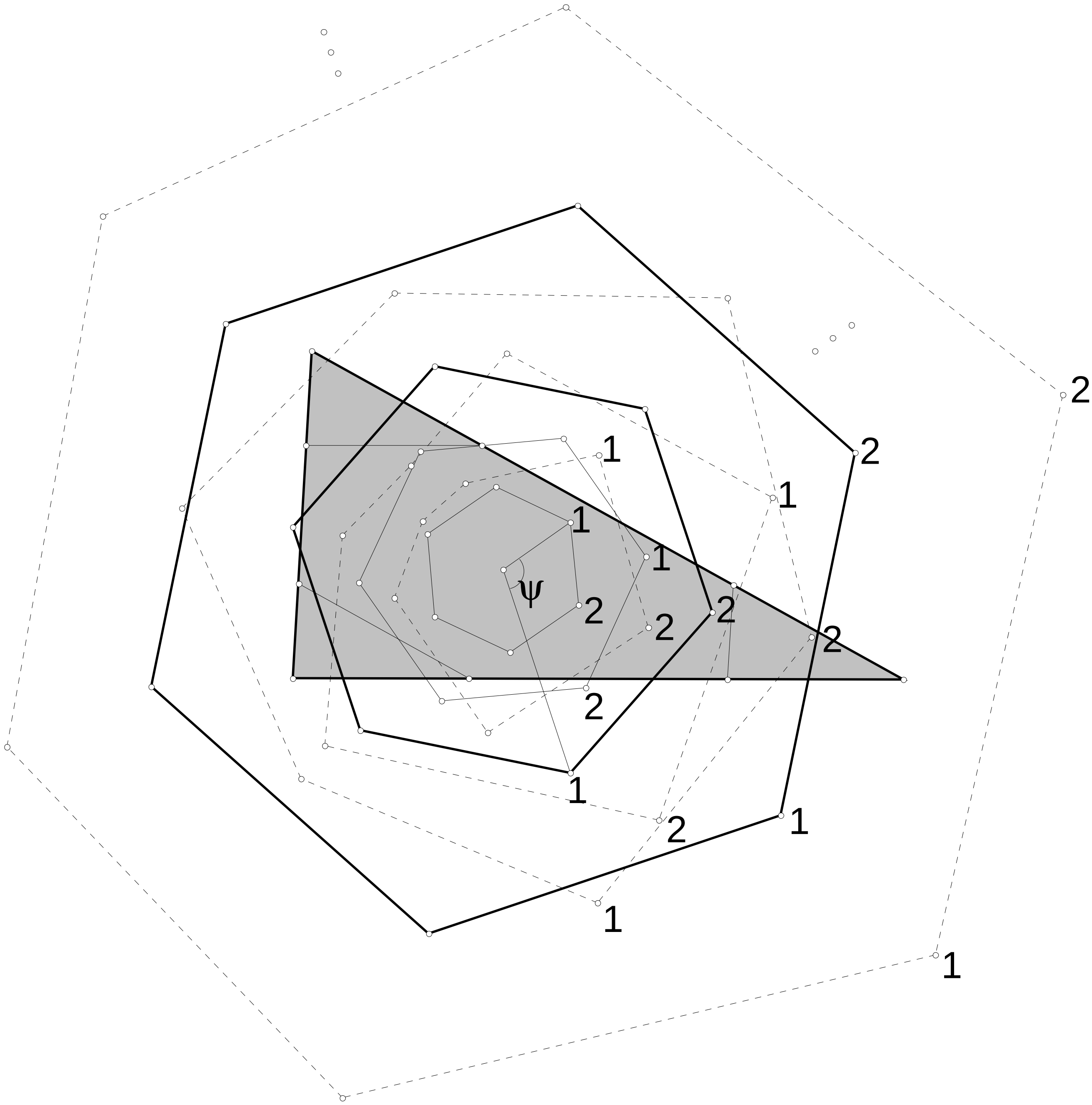

FIGURE 2

First two elements in sequences ***H*** (bold) and ***h*** (light).

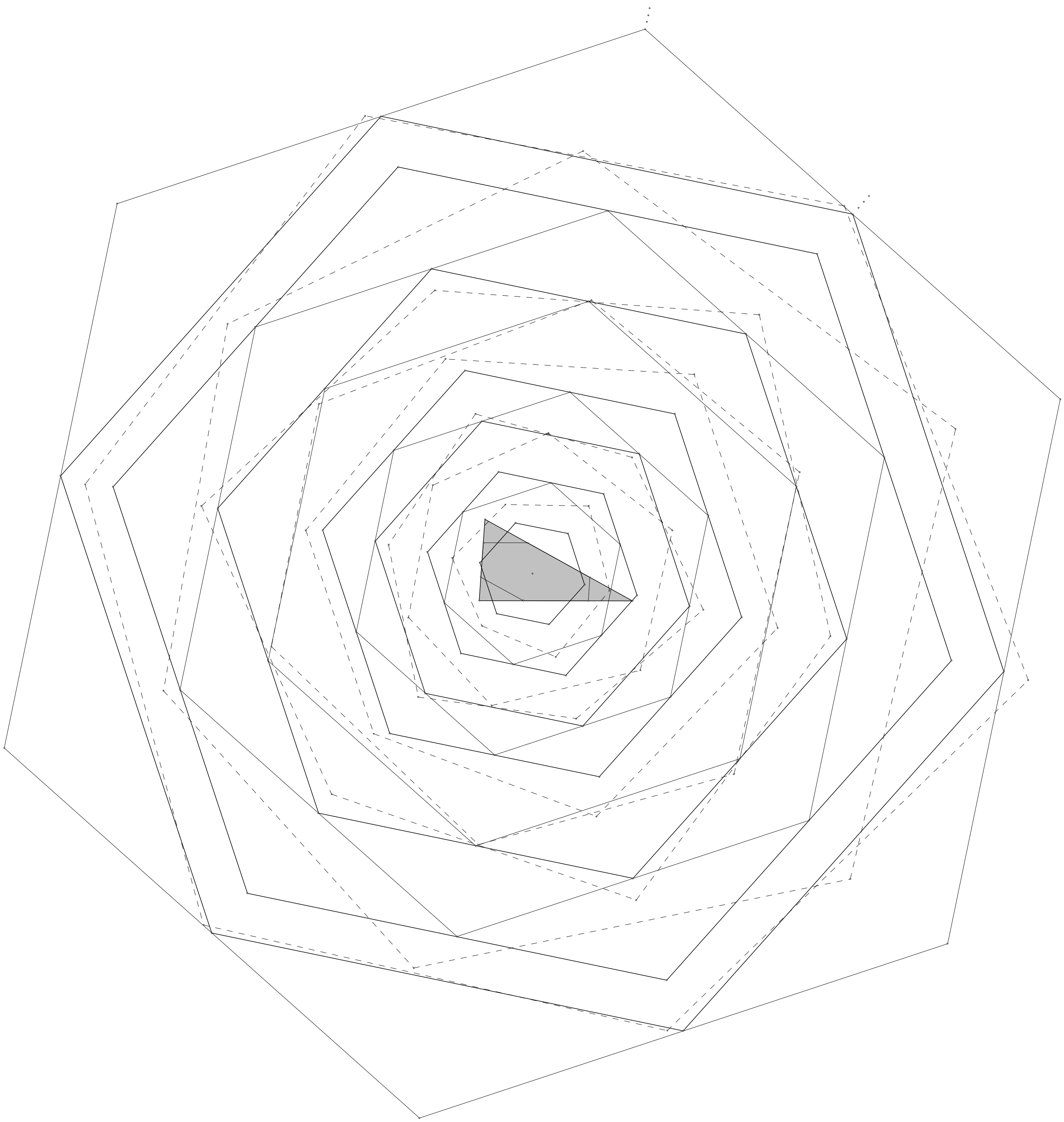

FIGURE 3

First seven elements in sequence $\boldsymbol{S}$ (bold) and first five in $\boldsymbol{S}_\tau$ (light).

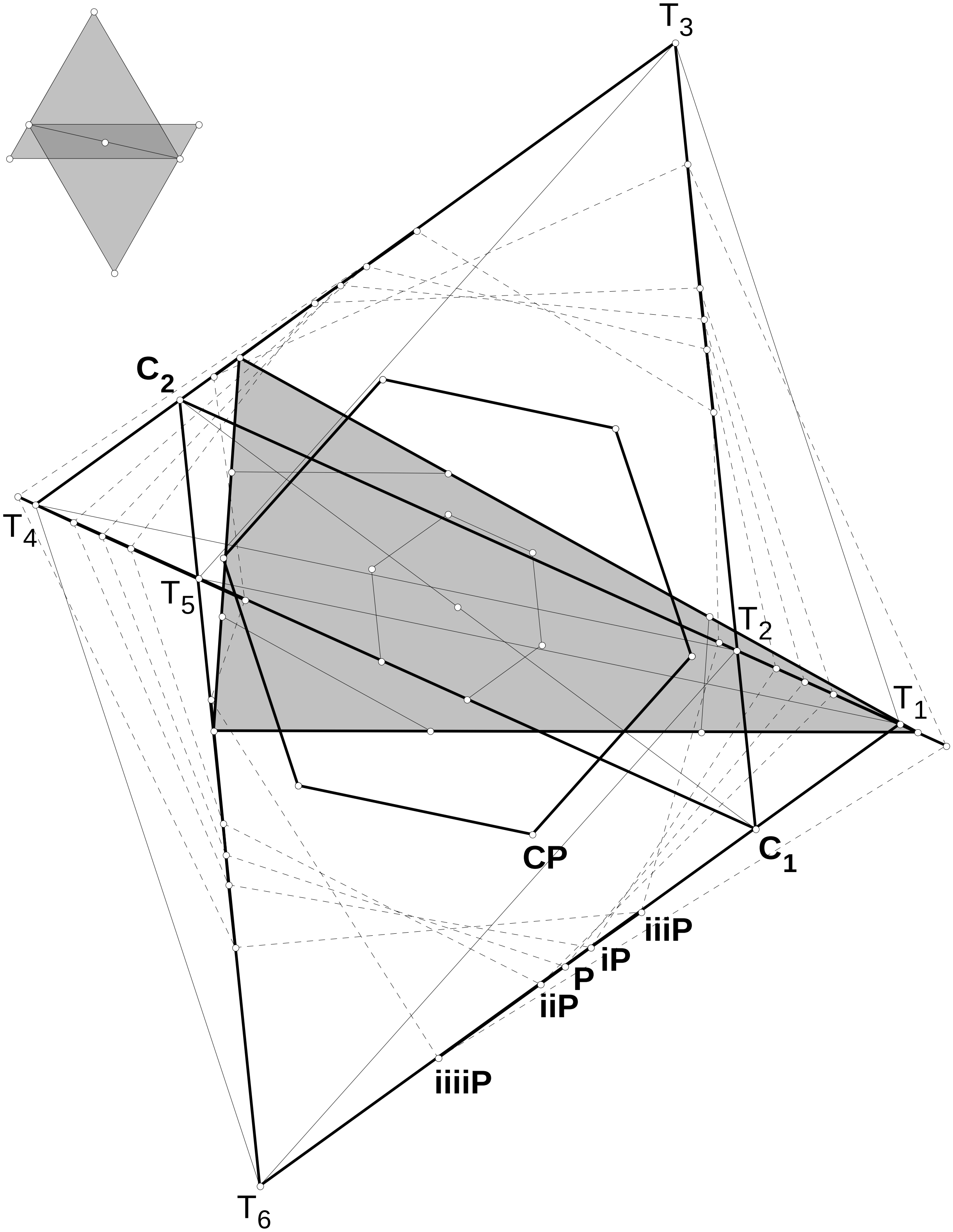

FIGURE 4

Identity cluster generating hexagons near $\sigma = \frac{1}{3}$. Sceptre motif at upper left.

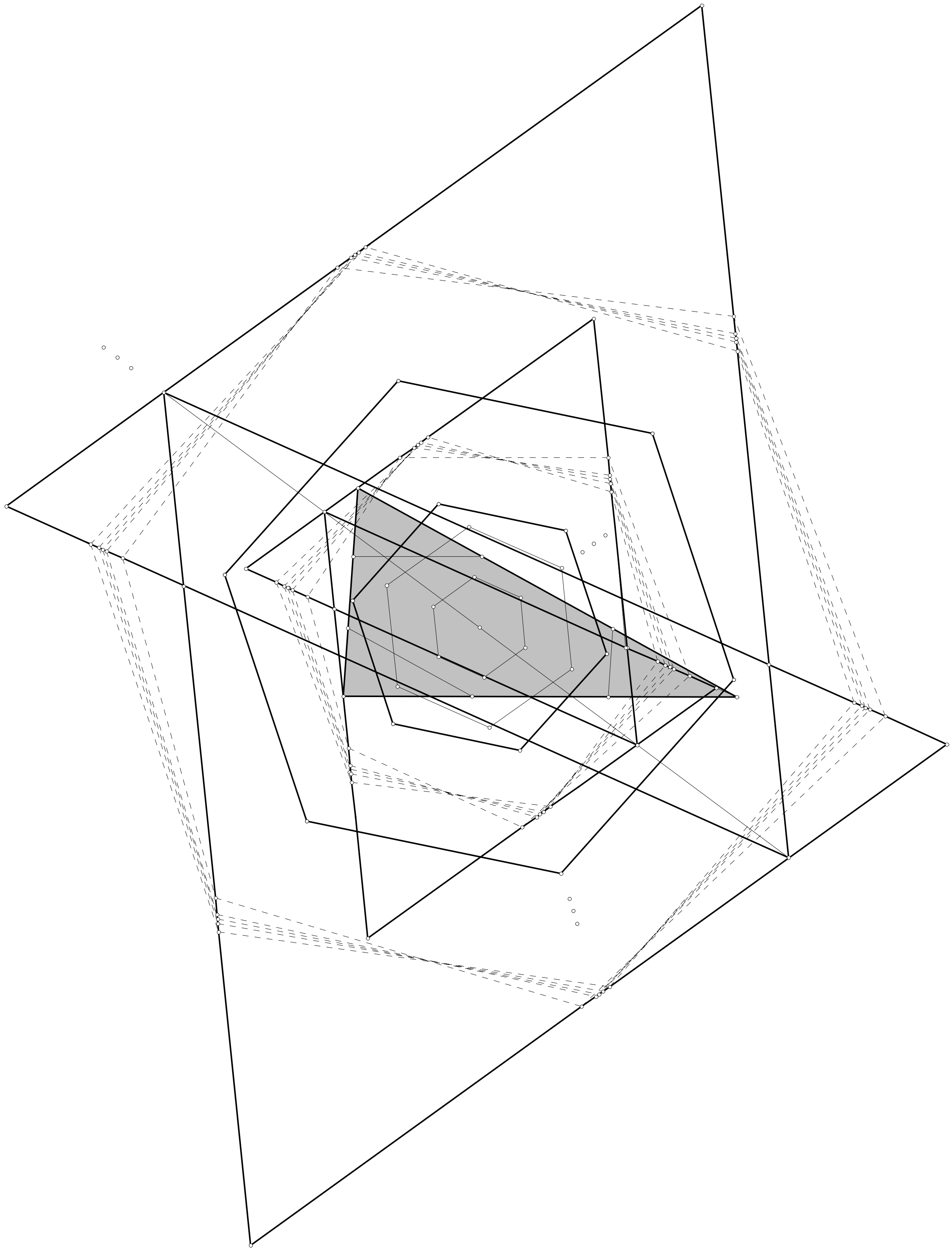

FIGURE 5

First two elements of sequence $\Psi(\boldsymbol{G})$ of equilateral triangle pairs (sceptres) parallel $\boldsymbol{g}$.

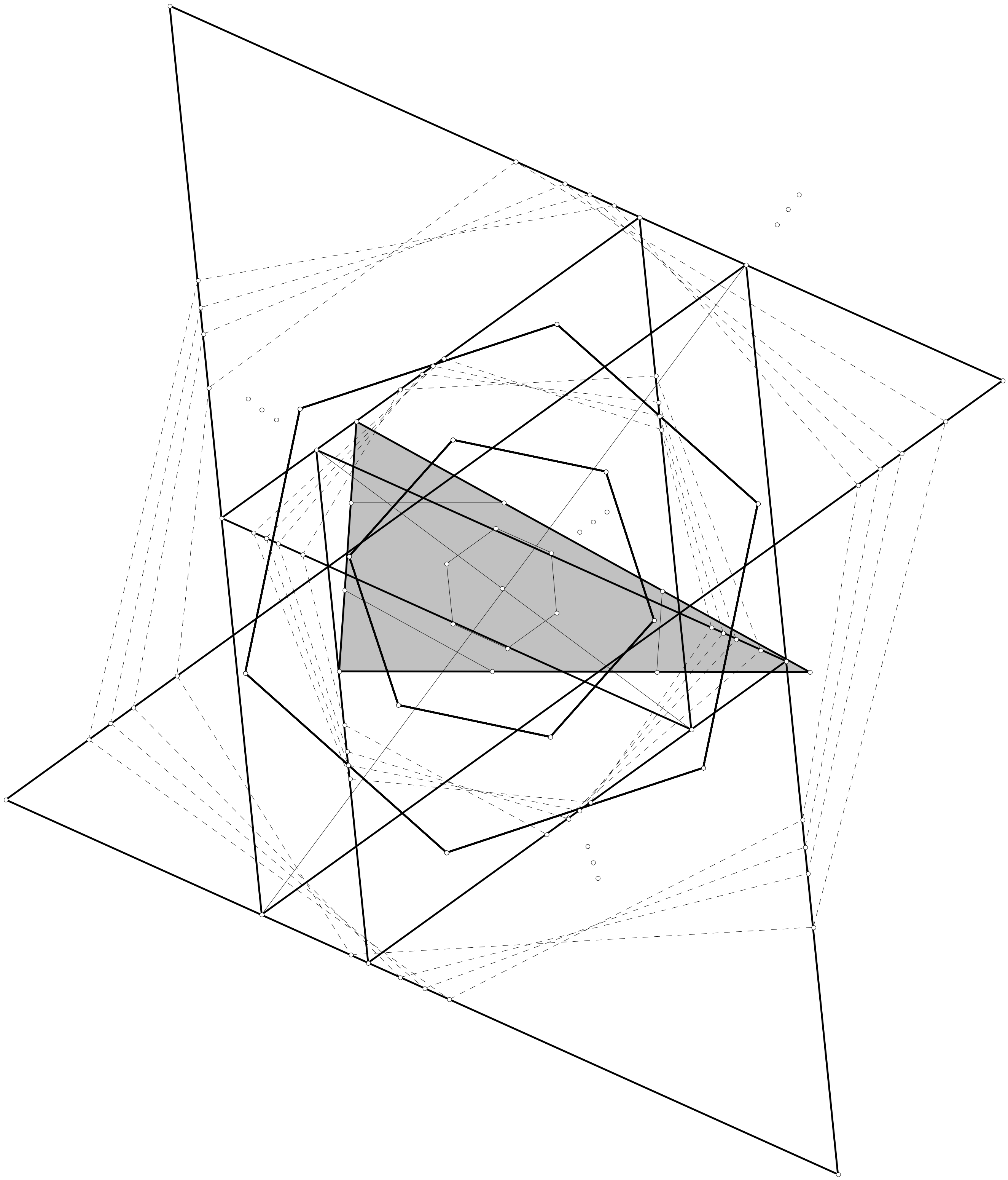

FIGURE 6

First two elements of sequence $\Psi(\boldsymbol{H})$ of equilateral triangle pairs (sceptres) parallel $\boldsymbol{g}$.