Eigenpolygon Decomposition of Polygons

Technical Memo 19

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Introduction

Eigenvectors and eigenvalues play powerful roles in linear algebra. A remarkable example of their use in geometry is presented here: Any n-gon can be represented as a complex linear sum of n eigenvectors. Since these eigenvectors are themselves n-gons, I call them *eigenpolygons*[1]. For example, any hexagon can be represented by a linear sum of six eigenhexagons, or, as the title suggests, it can be decomposed into six eigenhexagons. And any hexagon can be decomposed into a sum of the same six eigenhexagons—hence the "eigen". The rightmost column of Figure 1 shows these characteristic, or fundamental, hexagons. The columns of this figure, in fact, are the eigenpolygons for triangles, quadrilaterals, pentagons, and hexagons, respectively. Details of the eigenpolygon decomposition of 2-dimensional polygons are presented below.

I became acquainted with this technique in [1] while searching for proof techniques to apply in my paper [2]. I commented on its beauty and efficacy to author Professor Tom Sederberg who informed me that an editor of his book might have contributed it! I found it very useful in completing [2]. Upon quizzing my computer graphics research colleagues, I discovered they were as unaware of the result as I. Hence this paper.

Eigenpolygons

As shown in [1, 2], there are interesting polygon operators that take n-gons to n-gons and which are linear. A famous one is the "Napoleon" operator that takes an arbitrary triangle to an equilateral triangle. Construct an equilateral triangle outward on each of the three sides of the given triangle. Connect the three centroids of these equilaterals. Napoleon's Theorem guarantees that the new triangle so formed is always equilateral. ([2] is a generalization of this to infinite sequences of hexagons and equilaterals on an arbitrary triangle.)

We shall take an n-gon to be embedded in the complex plane. Hence an n-gon is an n-tuple of points in the complex plane, in traversal order. So a linear operator on polygons is just an ordinary linear operator on points in the complex plane. A linear operator on a triangle, for example, is a 3x3 matrix of complex constants applied to a 3-element vector representing the polygon. In general, $\mathbf{p'} = \mathbf{Mp}$ represents the operator $\mathbf{M}$ working on polygon $\mathbf{p}$ to form new polygon $\mathbf{p'}$,

[1] Tony de Rose and Ed Catmull of Pixar have proposed the term "basis polygons" instead as more intuitive and less bizarre.

where **M** is an *n*x*n* matrix and **p** and **p'** are *n*-element vectors. I use column vectors in this presentation.

If there is a scalar *s* such that linear operator **M** applied to a vector **v** is **Mv** = *s***v**, then *s* is called an *eigenvalue* of operator **M**. Any nonzero[2] vector **e** such that **Me** = *s***e** is an *eigenvector* for eigenvalue *s* (for operator **M**). For instance, if *s* is real, then **M** simply scales **e** along itself. For another, if *s* is a pure rotation e^{iq}, then **M** simply rotates **e** by angle q. In either case, **M** can be an elaborate operator, but it simplifies along its eigenvectors. Examples of elaborate operators **M** are presented below.

If an arbitrary polygon (vector) **p** can be written as a sum of eigenvectors, $\mathbf{p} = \sum a_k \mathbf{e}^{(k)}$, with the a_k complex scalars, then the general case simplifies too, to $\mathbf{p}' = \mathbf{Mp} = \sum a_k l_k \mathbf{e}^{(k)}$, where eigenvalue l_k corresponds to eigenvector $\mathbf{e}^{(k)}$. We are interested here in operators **M** for which the $\mathbf{e}^{(k)}$ are the eigenpolygons shown in Figure 1 (and their generalization to arbitrary *n*). The eigenvalues for these particular eigenvectors are discussed next.

Let w be the complex operator that rotates a complex point by angle $\frac{2p}{n}$ counterclockwise about the origin—ie, $w = e^{2pi/n}$. The powers of w are therefore the n^{th} roots of unity. These are exactly the eigenvalues we are interested in, and the corresponding eigenvectors, the eigenpolygons, are the vectors $\mathbf{e}^{(k)} = [1 \quad w^k \quad w^{2k} \quad \cdots \quad w^{(n-1)k}]$. For example, the leftmost column in Figure 1 contains the three eigentriangles $\mathbf{e}^{(1)} = [1 \quad w \quad w^2]$, $\mathbf{e}^{(2)} = [1 \quad w^2 \quad w^4]$ $= [1 \quad w^2 \quad w]$, and $\mathbf{e}^{(3)} = [1 \quad w^3 \quad w^6] = [1 \quad 1 \quad 1]$, a degenerate triangle. In general, there are *n* eigenpolygons and *n* eigenvalues for *n*-gons.

Figure 2 shows a given hexagon **h**, where, as usual, the unmarked point is the origin. It also shows the 6-tuple **a** whose elements are the coefficients in the eigenpolygon decomposition of **h**. The six eigenhexagons are repeated from Figure 1 for visual convenience. Notice that the triangles are really hexagons because they are traversed twice. Similarly the line segment is a hexagon because it is traversed roundtrip three times, and the point is a degenerate hexagon. So $\mathbf{h} = \sum_{k=1}^{6} a_k \mathbf{e}^{(k)} = \mathbf{aE}$, where matrix **E** which has $\mathbf{e}^{(k)}$ as it k^{th} row, or $h_k = \mathbf{a} \cdot \mathbf{E}^{(k)}$, where $\mathbf{E}^{(k)}$ is the k^{th} column of **E**. In words, each vertex of **h** is the dot product of **a** (itself a hexagon) and the vector formed from the corresponding vertices of the eigenhexagons. For example, h_2 is the dot product of **a** and the vector of vertices labeled 2 in the list of eigenhexagons of Figure 2. And h_3 is the dot product of the same **a** but with the vector of vertices labeled 3. But perhaps the simplest interpretation is taken directly from $\mathbf{h} = \sum_{k=1}^{6} a_k \mathbf{e}^{(k)}$—a linear sum of the eigenpolygons.

[2] The zero vector **0** is always an "eigenvector" of "eigenvalue" 0 for any operator **M**—ie, **M0** = 0**0**—so it is defined away.

This is the precise meaning of the concept, in the Introduction and title, of a decomposition of an n-gon into eigenpolygons.

How to Compute an Eigenpolygon Decomposition

This is simple. Since $\mathbf{p} = \mathbf{aE}$, then $\mathbf{a} = \mathbf{pE}^{-1}$, assuming the inverse exists. For convenience, $\mathbf{E}$ and $\mathbf{E}^{-1}$ are stated for the four cases of Figure 1:

n, w	3, $e^{2\pi i/3}$	4, $e^{\pi i/2}$	5, $e^{2\pi i/5}$	6, $e^{\pi i/3}$
$\mathbf{E}$	$\begin{bmatrix} 1 & w & w^2 \\ 1 & w^2 & w \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & w & -1 & w^3 \\ 1 & -1 & 1 & -1 \\ 1 & w^3 & -1 & w \\ 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & w & w^2 & w^3 & w^4 \\ 1 & w^2 & w^4 & w & w^3 \\ 1 & w^3 & w & w^4 & w^2 \\ 1 & w^4 & w^3 & w^2 & w \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & w & w^2 & -1 & w^4 & w^5 \\ 1 & w^2 & w^4 & 1 & w^2 & w^4 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & w^4 & w^2 & 1 & w^4 & w^2 \\ 1 & w^5 & w^4 & -1 & w^2 & w \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$
$\mathbf{E}^{-1}$	$\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ w^2 & w & 1 \\ w & w^2 & 1 \end{bmatrix}$	$\frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ w^3 & -1 & w & 1 \\ -1 & 1 & -1 & 1 \\ w & -1 & w^3 & 1 \end{bmatrix}$	$\frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ w^4 & w^3 & w^2 & w & 1 \\ w^3 & w & w^4 & w^2 & 1 \\ w^2 & w^4 & w & w^3 & 1 \\ w & w^2 & w^3 & w^4 & 1 \end{bmatrix}$	$\frac{1}{6}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ w^5 & w^4 & -1 & w^2 & w & 1 \\ w^4 & w^2 & 1 & w^4 & w^2 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ w^2 & w^4 & 1 & w^2 & w^4 & 1 \\ w & w^2 & -1 & w^4 & w^5 & 1 \end{bmatrix}$

Do you see the pattern? In the case n = 3, consider the 2x2 submatrix obtained by deleting the row and column of all 1s. This submatrix of $\mathbf{E}^{-1}$ is the left-to-right (or top-to-bottom) reflection of the submatrix of $\mathbf{E}$. This holds for all n. Another way to say this is: The columns of $n\mathbf{E}^{-1}$ are (a permutation of) the rows of $\mathbf{E}$—ie, the eigenpolygons.

Figure 2 shows each of the eigenhexagons weighted by the appropriate a_k for the decomposition of the given hexagon $\mathbf{h} = \begin{bmatrix} 1 & i & -2+2i & -3 & 2-2i & 1-i \end{bmatrix}$. It also shows the partial sums as the weighted eigenhexagons are added together. The principal insight gained from this figure is that the contribution of the degenerate eigenhexagon $\mathbf{e}^{(6)}$ is to offset the centroid of the given polygon from the origin. In other words, $a_6 = 0$ for hexagons centered on the origin, and similarly for all n-gons.

Operators with Eigenpolygons as Eigenvectors

Although the purpose of this memo is to present the eigenpolygon decomposition of polygons, it might be of additional interest to know just which linear operators $\mathbf{M}$ benefit from this decomposition—ie, which $\mathbf{M}$ have the eigenpolygons as eigenvectors and hence offer the computational simplification $\mathbf{p}' = \mathbf{Mp} = \sum a_k \lambda_k \mathbf{e}^{(k)}$ discussed above.

Consider the *shift* (*backwards*) *operator* $\mathbf{S}$ defined for n-gon $\mathbf{p}$ by $\mathbf{p}' = \mathbf{Sp}$ defined by $p'_k = p_{k+1}$, where the indices wrap modulo n. For example, $\mathbf{S}$ yields $p'_6 = p_1$ for hexagons. The complement $\overline{\mathbf{S}}$ is similarly defined by $p'_k = p_{k-1}$. It is not difficult to show that w^k, the n^{th} roots of unity, are the eigenvalues of **S** with eigenvectors $\mathbf{e}^{(k)}$. Take quadrilaterals for instance:

$$\mathbf{S}\mathbf{e}^{(k)} = \mathbf{S}\begin{bmatrix}1 & w^k & w^{2k} & w^{3k}\end{bmatrix} = \begin{bmatrix}w^k & w^{2k} & w^{3k} & 1\end{bmatrix} = w^k\mathbf{e}^{(k)}.$$

Similarly, the conjugates $\overline{w}^k$ are the eigenvalues of $\overline{\mathbf{S}}$ with the same eigenvectors.

Any operator that can be expressed as a polynomial in **S** and $\overline{\mathbf{S}}$ is one of the **M** that benefit from eigenpolygon decomposition. The Napoleon operator **N** that satisfies Napoleon's Theorem is, not surprisingly, one of these. It can be expressed by $\mathbf{N} = c\mathbf{S} + (1-c)\mathbf{I}$, where **I** is the identity operator mapping an n-gon to itself and $c = \frac{1}{3}\sqrt{3}e^{-pi/6}$ (see [1]). In words, **N** constructs an isosceles triangle on each side of a given triangle with matching angles of $\frac{p}{6}$, which is equivalent to constructing equilateral triangles on each side and connecting their centroids. Since **S** has eigenvalues w^k, for $w = e^{2pi/3}$, and **I** has eigenvalue 1, the eigenvalues of **N** are $l_k = cw^k + (1-c)$. For arbitrary triangle **t** and its eigentriangle decomposition $\mathbf{t} = \sum_{k=1}^{3} a_k\mathbf{e}^{(k)}$, the streamlined Napoleon calculation becomes $\mathbf{t}' = \mathbf{N}\mathbf{t} = \sum b_k\mathbf{e}^{(k)}$, where $b_k = a_k l_k$.

It is easy to describe, using the eigentriangle decomposition of triangles, why the Napoleon Theorem works: One of the equilateral eigentriangles is annihilated by the operator leaving only the other which is, of course, regular and hence the (outer) Napoleon triangle. In particular, notice that $l_2 = 0$.

The beautiful Douglas-Neumann Theorem generalizes the Napoleon Theorem to arbitrary n-gons. Chang and Sederberg [1] give an excellent presentation of this theorem which states that a particular series of constructions of isosceles triangles on the sides of a given n-gon must reduce it to a regular n-gon. This can be expressed by a sequence of operators, each of which has the eigenpolygons as eigenvectors. Each operator in the sequence annihilates one of the eigenpolygons in the decomposition, ultimately leaving only one.

Finally, in [2], I generalize Napoleon's Theorem in another direction, showing that an arbitrary triangle operated on by sequences of operators from a certain class always maps to an infinite sequence of hexagons (equilateral triangles in a special case) that are all concentric with one another and the given triangle and parallel to one another. As you might guess, each operator in the class has the eigenpolygons as eigenvectors, and I use eigen polygon decomposition to prove the main theorems.

References

1. G Chang and T W Sederberg, *Over and Over Again*, Mathematical Association of America, Washington, DC, 1997.

2. Smith, Alvy Ray, *Hexagon Sequences on a Triangle*, manuscript to be submitted for publication, Microsoft, Redmond, WA, Oct 1998.

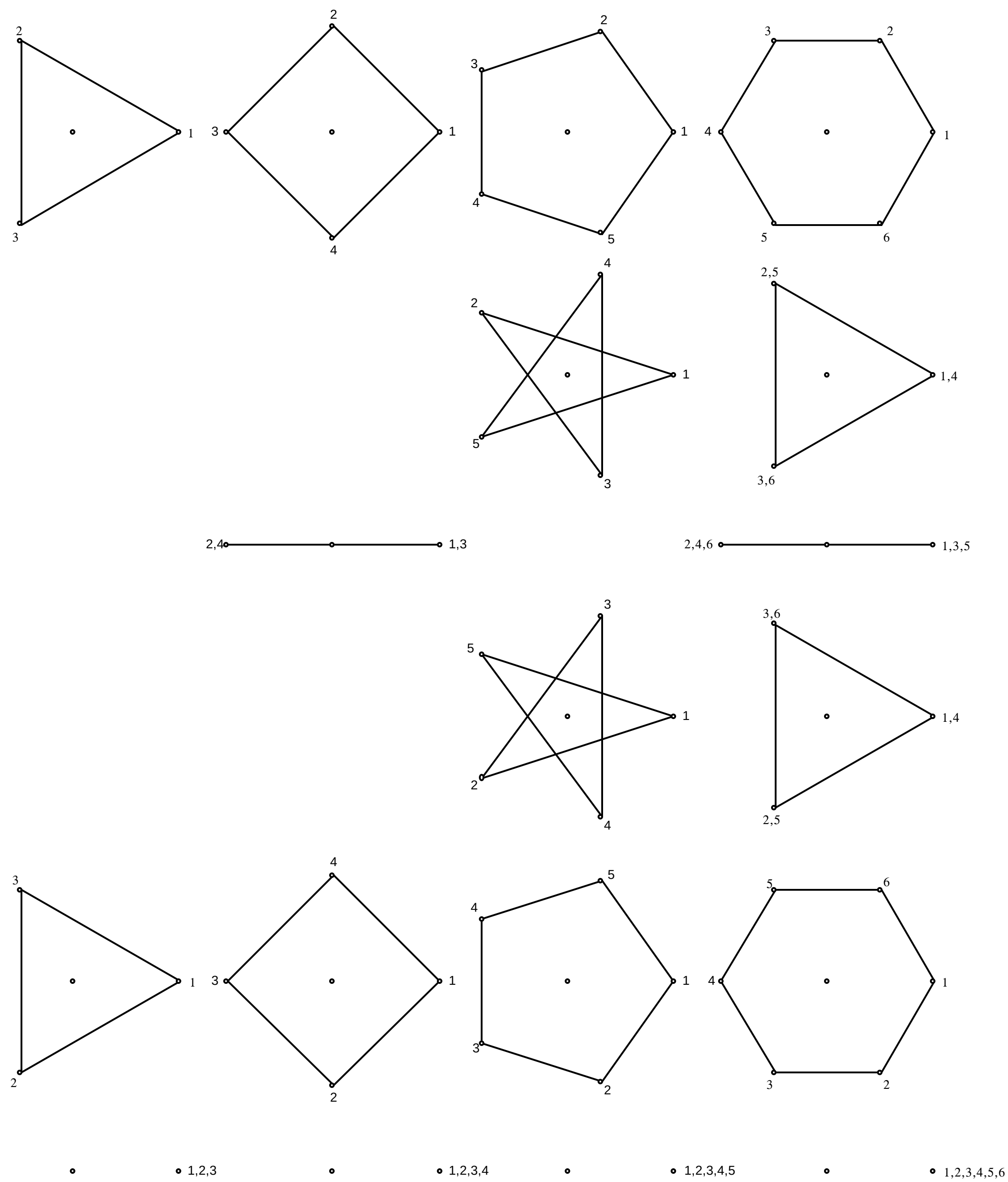

Figure 1
The *n* eigenpolygons for *n*-gons, *n* = 3, 4, 5, and 6 (in columns, left to right).

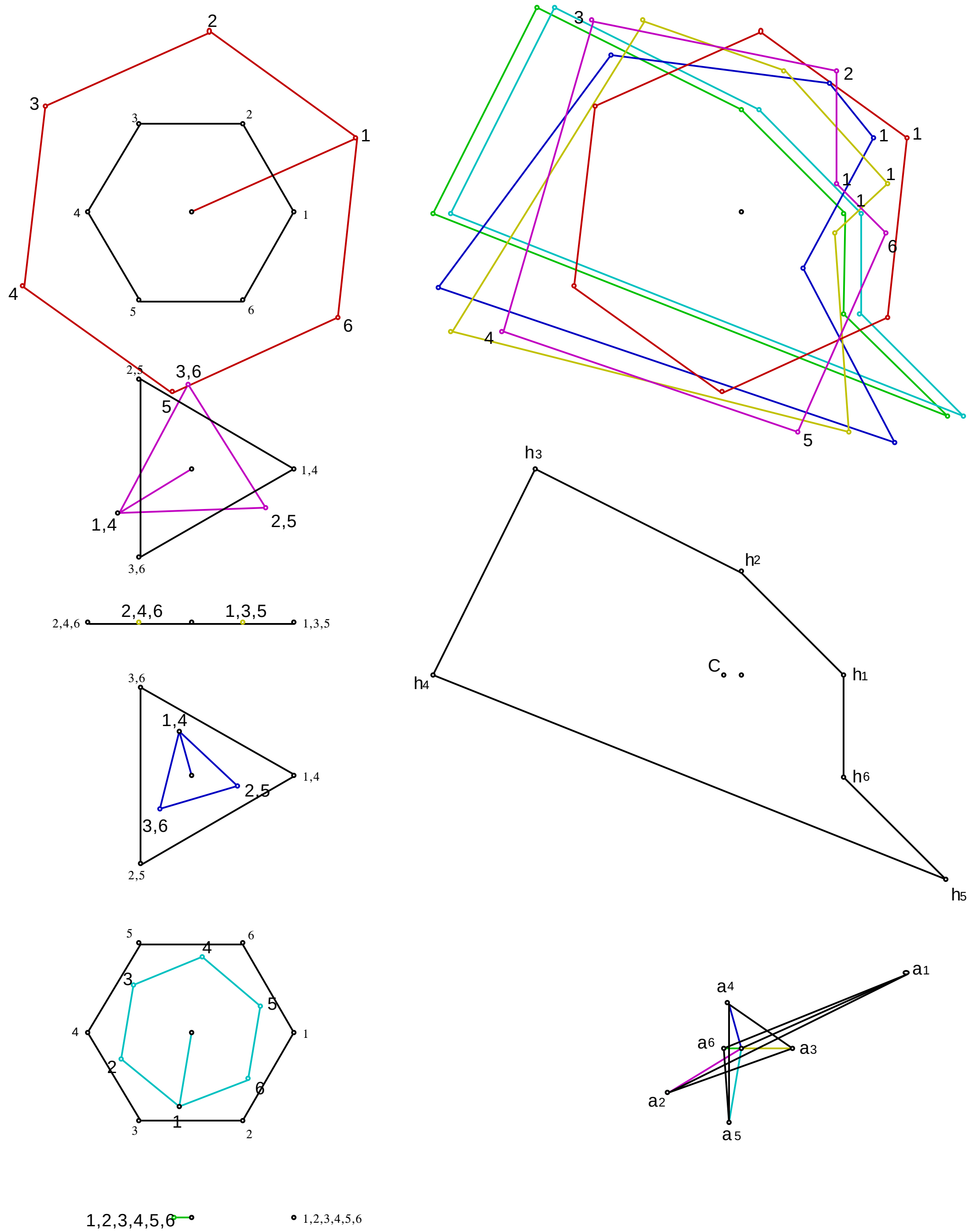

Figure 2

Hexagon *h* and its decomposition *a* into eigenhexagons. *C* is its centroid. The eigenhexagons on the left are shown multiplied by the a_k. The upper right shows the intermediate steps of the eigenhexagon addition that yields *h*. a_6 is seen to be the offset of the centroid *C* from the origin.